Rocket Design for a Specified Trajectory

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The equation of motion for the powered flight of a particular stage of a multistage vehicle is reduced to a first-order linear differential equation in the vehicle weight, whose coefficients are known functions of time, by the assumption that the trajectory is known. The trajectory is arbitrary and includes lift. The design parameters include total weight, payload weight, specific impulse, propellant mass fraction, and vehicle diameter. Furthermore, it is possible to include specified constraints on the vehicle such as fineness ratio or burning rate of solid propellants. This method is based on certain trajectory constants determined from integrations over the known powered trajectory. With the exception of specific impulse, a simple relation can be formed (no further integration is necessary) for comparing changes in vehicle weight with changes in other design parameters. A method for choosing the best propellant and chamber pressure from a given set of propellants is discussed. An example of determining the optimum chamber pressure for the sustainer stage of a two-stage vehicle is also presented.

Nomenclature

weight = value at sea level
$\mathbf{a}$ = acceleration in inertial space
a_u = projection of $\mathbf{a}$ onto $\mathbf{u}$
B = ballistic coefficient at burnout W_T/S
b = web thickness of propellant grain
b^* = effective web thickness for radial-burning grains $(D_{G1} - D_{h1})\, b/2b_1$
C_A = axial-force coefficient $C_D \cos\alpha - C_L \sin\alpha$
C_D = drag coefficient
C_L = lift coefficient
C^* = characteristic velocity of the propellant
D = external diameter of the rocket motor
D_G = average external diameter of the propellant grain
D_h = $(4S_h/\pi)^{1/2}$
E = trajectory constant for known I_s [see Eq. (7)]
E' = value for E at time t
F = thrust $\dot{W}_p I_s$ or $-\dot{W} I_s$
f = fineness ratio $(L_P + L_R)/D$
G = a known function of time $C_A q/I_s$
g = acceleration due to gravity at time t
g_0 = acceleration due to gravity at sea level
H = a known design constant S_h/S_t
I_s = specific impulse, a known function of time
$\bar{I}_s$ = specific impulse at a specified altitude
J = a known function of time defined by Eq. (4)
k = $(D - D_G)/2D$
k' = $(1 - 2k)$
K = trajectory constant for known λ and I_s [see Eq. (12)]
$\bar{K}$ = trajectory constant for known λ and $\bar{I}_s$ [see Eq. (12)]
K_G = trajectory constant for known λ, I_s, and b for end-burning grains [see Eq. (16)]
K_B = trajectory constant for known λ, I_s, and B [see Eq. (28)]
L_P = payload length
L_R = total rocket motor length
L_c = length of the combustion chamber
n = ratio of nozzle weight to W_i
P = payload weight (may include structure and fins)
P_c = chamber pressure
Q = trajectory constant for known I_s [see Eq. (6)]
Q' = value for Q at time t
q = dynamic pressure
$\bar{r}$ = average burning rate b/T
S = reference area $\pi D^2/4$
S_G = $\pi D_G^2/4$
S_h = average port area for radial burning propellants
S_t = nozzle throat area
T = burning time
t = arbitrary time between ignition ($t = 0$) and burnout ($t = T$)
t_c = thickness ratio of the combustion chamber D/L_c
$\mathbf{u}$ = unit vector in direction of longitudinal vehicle axis
W = vehicle weight at time t
W_0 = vehicle weight at time $t = 0$, $W_0 = P + W_R$
$\dot{W}$ = $-\dot{W}_p$
W_p = propellant weight at time $t = 0$
$\dot{W}_p$ = propellant flow rate at time t
W_R = rocket motor weight at time $t = 0$
W_i = rocket motor inert weight $W_R - W_p$
α = angle of attack, a known function of time
δ = $(I_{s1} - I_s)/I_s$
λ = propellant mass fraction W_p/W_R
ρ_P = effective payload density based on a cylindrical volume
ρ_R = effective motor density $W_R/L_R S$
ρ_G = weight density of a solid propellant grain
ϕ = angle between $\mathbf{u}$ and vertical

Subscripts

T = value at time T
1 = value for design 1

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Introduction

THE preliminary goal in most rocket design problems is to obtain the trajectory (or position-time function) that best satisfies the performance constraints of a specific mission. The ultimate goal, and the problem discussed here, is to obtain this trajectory with a rocket vehicle subject to given constraints on the design parameters. The vehicle designer often finds that design changes in the payload (or perhaps propellant) are required after the trajectory is established. The relations presented in this paper afford the designer a direct method for determining alternate vehicle designs that accommodate such changes with the constancy of the trajectory guaranteed.

This method is based on certain trajectory constants that result from integrations over the desired trajectory. Except for cases in which specific impulse is changed, no further integrations are necessary to determine new vehicle designs. When specific impulse is changed, the integrations are quite simple in nature and do not require the complex and costly solutions of most computer flight programs. New designs of a multistage vehicle are determined by considering the design changes in the last stage first, and using the new value for the last stage weight as the payload for the next-to-last stage, etc.

The initial vehicle design that determines the desired trajectory will be called design 1, and the new design will be called design 2. The items of importance are the following three functions of time: 1) the position of the center of mass, 2) the orientation of the longitudinal body axis, and 3) the axial-force coefficient C_A in the direction of the longitudinal body axis. The first two assumptions are that the time functions of items 1 and 2 will remain unchanged between the two designs during powered flight. Item 2 requires that the angle-of-attack/time history remain unchanged; for a constant trajectory, the angle-of-attack function $\alpha(t)$ should not change appreciably. If C_A changes by a given percentage between designs, then the third assumption requires that this percentage be independent of Mach number. The coefficient C_A does include the induced drag due to α and control surface deflections of a lifting vehicle,[1] and is equal to $C_D \cos\alpha - C_L \sin\alpha$. The last two assumptions are reasonable as long as the two designs are of the same order of magnitude.

Unless the ballistic coefficient of design 1 at burnout is specified as a configuration constraint for design 2, the second design is not expected to precisely follow the original position-time function after burnout. In general, design 2 can be controlled to fly the same path after burnout, but the proper value for thrust to maintain the desired velocity-time function can no longer be selected. It will be shown later that the ballistic coefficient at burnout for rocket motors using the same end-burning solid propellant and chamber pressure does not change with changes in payload weight. For rockets that burn out at high altitudes (e.g., most sounding rockets), the subsequent trajectory is a slow function of ballistic coefficient. For low-altitude rockets (e.g., air-to-air or ground-to-air intercept missiles), using the same radial-burning propellant and chamber pressure, the ballistic coefficient at burnout and subsequent velocity-time function will increase with increasing payload weight.

Equation of Motion

The unit vector $\mathbf{u}$ is directed forward along the longitudinal axis of the vehicle and is used to designate this axis. The equation of motion for the center of mass in the direction of the axis $\mathbf{u}$ is

$$F - C_A q S - gW \cos\phi / g_0 = W a_u / g_0 \tag{1}$$

For cases where the thrust $-\dot{W} I_s$ is not directed along $\mathbf{u}$, it can be corrected by a suitable time function. The angle ϕ is taken between local vertical and $\mathbf{u}$, and a_u is the projection of $\mathbf{a}$ onto $\mathbf{u}$. The axial aerodynamic force[1] $C_A q S$ is taken positive in the direction $-\mathbf{u}$. Equation (1) can be written in the form

$$\dot{W} + GW = -JS \tag{2}$$

where

$$J = C_A q / I_s \tag{3}$$

$$G = (a_u + g \cos\phi)/(g_0 I_s) = (F - C_A q S)/(W I_s) \tag{4}$$

With the trajectory known, q and g are known functions of time. Since $C_A(t)$ and $\mathbf{u}(t)$ are assumed to be known, a_u and ϕ are also known functions. The specific impulse (I_s) is dependent on the choice of propellant, chamber pressure (P_c), and altitude, which is a known time function. Thus, a choice of propellant and P_c will result in known functions $G(t)$ and $J(t)$. Equation (2) can now be considered a first-order linear differential equation in W, the sea level vehicle weight.

For a chosen I_s, the solution to Eq. (2) in terms of the initial weight W_0 and the weight W_T at burnout is

$$W_T = W_0/E - QS \tag{5}$$

where

$$Q = \frac{1}{E} \int_0^T E' J dt = \text{const} \tag{6}$$

$$E = \exp\left(\int_0^T G dt\right) = \text{const} \tag{7}$$

$$E' = \exp\left(\int_0^t G dt\right) \tag{8}$$

The trajectory constants for a chosen I_s are defined as E and Q.

For the general case, E and Q are determined by numerical integration. It should be pointed out that if C_A in Eq. (3) is to be changed, after the integration for Q, by a given percentage that is independent of Mach number, then Q will also change by this percentage.

Flow Rate and Mass Fraction

Marked variations in $\dot{W}_p$ from its average value must necessarily affect $I_s(t)$ due to changes in chamber pressure and nozzle efficiency. These variations should be included in the initial flight of design 1. From the differential equation (2) and its solution, the rate of propellant discharge at an arbitrary time t on the trajectory is found to be

$$\dot{W}_p = -\dot{W} = GW_0/E' + (J - GQ')S \tag{9}$$

where

$$Q' = \frac{1}{E'} \int_0^t E' J dt$$

It is evident from this equation that design 2 will not necessarily have the same "shape" for $\dot{W}_p$ vs time as design 1, if design 2 is to precisely follow the specified position-time function. However, computer solutions have shown that, when design 2 is flown with the same $\dot{W}_p$ vs time "shape," the resulting trajectory varies only a few percent from the original when the two designs are of the same order of magnitude (see Table 1). Equation (9) shows that the change in "shape" for $\dot{W}_p$ is related to the change in W_0/S, the ballistic coefficient at ignition.

The vehicle weight at burnout is

$$W_T = P + W_R - W_p = P + (W_0 - P)(1 - \lambda) \tag{10}$$

Substitution of this result into Eq. (5) shows

$$W_0 = \frac{\lambda}{E^{-1} - (1 - \lambda)} P + \frac{Q}{E^{-1} - (1 - \lambda)} S \tag{11}$$

$$\equiv KP + \bar{K}S \tag{12}$$

where K and $\bar{K}$ are trajectory constants for known I_s and propellant mass fraction ($\lambda = W_p/W_R$). It is apparent that λ must be larger than $1 - E^{-1}$ for a finite solution.

Solid and Liquid Propellants

The preceding trajectory constants are dependent on the chosen I_s. For a solid propellant, I_s and burning rate are related to P_c. Therefore, if I_s is chosen for a known propellant, the burning rate must also be known. The web thickness b of the propellant is determined as the integral of the burning rate over the known burning time T. The average burning rate is $\dot{r} = b/T$.

For a given P, a choice of S in Eq. (12) will result in a rocket motor weight of $(K - 1)P + \bar{K}S$ and a motor diameter of $(4S/\pi)^{1/2}$. For an *end-burning propellant*, the dimensions of the grain can be determined from λ, the motor wall thickness, and the propellant grain density ρ_G. If the resulting length of the propellant is not equal to the known value $b = \dot{r}T$, the wrong choice was made for S. The proper value for S can be determined by trial-and-error, but a new expression is desired which includes b as a constraint. The average motor wall thickness per unit diameter k, which includes thermal insulating material and propellant inhibitor, is assumed to be a known function of maximum chamber pressure. Therefore, the relation between average cross-sectional area of the propellant grain S_G and the motor area S is

$$S_G = k'^2 S \text{ or } D_G = k'D \tag{13}$$

where $k' \equiv (1 - 2k)$. For an end-burning propellant, b is the propellant length, and

$$b = \lambda W_R/\rho_G S_G = \lambda(W_0 - P)/(k'^2 \rho_G S) \tag{14}$$

Solving for S and substituting it into Eq. (12) gives

$$W_0 = K_G P \tag{15}$$

where K_G is the trajectory constant for given I_s, b, and λ, and it is determined by

$$K_G = (Kk'^2\rho_G b - \bar{K}\lambda)/(k'^2\rho_G b - \bar{K}\lambda) = \text{const} \tag{16}$$

It will be shown later that Eq. (16) is equivalent to constraining the ballistic coefficient of the vehicle at burnout. Therefore, all vehicle designs with the same K_G will have the same trajectory after burnout as well as before.

With a *radial-burning propellant*, the problem of constraining b for a given P is more complex. A choice for S will result in a known value for S_G in Eq. (13), and S_G is related to the average port area S_h by the volume-packing factor for the grain perforation geometry $(1 - S_h/S_G)$. For a specified grain perforation geometry (i.e., that of design 1), S_h/S_G is a known function of b/D_G, so that S is now a known function of S_h. The proper value for S can be obtained by a trial-and-error solution, which minimizes S_h without causing excessive erosion of the propellant. The following discussion presents an approximate solution for S, or at least a starting value for the preceding trial-and-error solution.

The effective port diameter D_h is related to S_h, D_G, and D by

$$D_h = (4S_h/\pi)^{1/2} = D_G - 2b^* = k'D - 2b^* \tag{17}$$

where b^* is the effective web thickness $(D_{G1} - D_{h1})b/2b_1$. The average port area can now be written as

$$S_h = \pi(D^2k'/4 - Dk'b^* + b^{*2}) \tag{18}$$

The propellant weight is

$$W_p = \lambda(W_0 - P) = \lambda(PK - P + \bar{K}S) \tag{19}$$

and the throat area S_t of the exhaust nozzle is related to the chamber pressure P_c by[2]

$$S_t = C^*\lambda(PK - P + \bar{K}S)/(g_0 P_c T) \tag{20}$$

where C^*, the known "characteristic velocity" of the propellant, is insensitive to P_c. It is important to insure that S_h is sufficiently large so that the gas velocity in the chamber is kept below a value that would erode the propellant grain. A parameter that is often used is

$$1.0 < H \equiv S_h/S_t < 2.5 \tag{21}$$

The value[3] of H is assumed to be known for a given "class" of motor designs. Equations (18, 20, and 21) can be used to form a quadratic in the motor diameter D:

$$D = \bar{b}/2\bar{a} + [(\bar{b}/2\bar{a})^2 - \bar{c}/\bar{a}]^{1/2} \tag{22}$$

where

$\bar{a} = (k'^2 - \bar{K}\bar{d})/4$ (no units)
$\bar{b} = k'b^*$ (length)
$\bar{c} = b^{*2} - (K - 1)P\bar{d}/\pi$ (length)^2
$\bar{d} = \lambda HC^*/(g_0 P_c T)$ (pressure)$^{-1}$

For *liquid propellants*, the designer may wish to establish the relation between W_0 and P under the constraint of a constant vehicle fineness ratio f, or ballistic coefficient at burnout B, so that they can be compared under a configuration constraint as well as a constraint on the trajectory.

For the *fineness ratio solution*, the payload diameter and D are assumed to be the same. A similar solution can be obtained when the payload diameter is a known value less than D (i.e., upper stages of known dimensions). The effective densities ρ_P and ρ_R of payload and rocket motor, respectively, are based on a cylindrical volume of cross section S, and include the volume packing factors due to the nose fairing of the payload and the actual shape of motor end closures and nozzle. The rocket motor length is

$$L_R = W_R/\rho_R S = [P(K - 1) + \bar{K}S]/\rho_R S \tag{23}$$

It should be remembered that ρ_R is dependent on the densities of the propellant and inert parts of the motor and the propellant mass fraction. The vehicle fineness ratio is

$$f = \frac{L_P + L_R}{D} = \frac{4P}{\rho_P \pi D^3} + \frac{4P(K - 1) + K\pi D^2}{\rho_R \pi D^3} \tag{24}$$

This can be reduced to a cubic in D as

$$D^3 - \bar{a}D^2 - \bar{b} = 0 \tag{25}$$

where

$$\bar{a} = \bar{K}/f\rho_R \qquad \bar{b} = (4P/\pi f)\{(1/\rho_P) + [(K - 1)/\rho_R]\}$$

Since D must be positive and the first-order term is missing, Eq. (25) is easily solved by use of the cubic equation:

$$D = (\bar{c} + \bar{d})^{1/3} + (\bar{c} - \bar{d})^{1/3} + \bar{a}/3 \tag{26}$$

where

$$\bar{c} = \bar{a}^3/27 + \bar{b}/2 \qquad \bar{d} = (\bar{c}^2 - \bar{a}^6/729)^{1/2}$$

With the solution for D, $S = \pi D^2/4$ and Eq. (12) determine W_0 for the new vehicle design.

The position and velocity at burnout are assumed to be known constants. Therefore, the *ballistic coefficient* can be used to compare various trajectories after burnout:

$$B \equiv W_T/S = W_0/ES - Q = (KP/S + \bar{K})/E - Q \tag{27}$$

This equation can be solved for W_0/P:

$$W_0/P = KE(B + Q)/[E(B + Q) - \bar{K}] \equiv K_B \tag{28}$$

which is the trajectory constant for a specified B, I_s, and λ. A comparison of this equation with Eq. (15) shows that a constraint on the chamber pressure and burning rate of an end-burning solid propellant is equivalent to constraining the ballistic coefficient at burnout. Though this result is not surprising, an equivalence statement between Eqs. (28) and (16) shows the precise relations involved.

Variations in Specific Impulse and Mass Fraction

With slight modification, most computer programs (no matter how simple or complicated) simulating rocket flight can be written to compute the fundamental trajectory parameters of each flight as an additional output of the program. With the subscript 1 denoting the vehicle used in the computer program (design 1), Eqs. (3) and (4) are used at each point in time to determine $J_1(t)$ and $G_1(t)$:

$$J_1 = C_A q/I_{s1} \quad \text{for} \quad I_{s1} \neq 0 \tag{29}$$

$$G_1 = (F_1 - C_A q S_1)/W_1 I_{s1} \quad \text{for} \quad I_{s1} \neq 0 \tag{30}$$

The trajectory parameters E_1 and Q_1 are then determined by integration in Eqs. (6–8). They are functions only of I_{s1} of design 1. If design 2 is to employ a different I_s-time function due to a different propellant or P_c, then values of G_1I_{s1}/I_s and J_1I_{s1}/I_s must be used to obtain E and Q for design 2. Therefore, the time functions G_1I_{s1} and J_1I_{s1} determined by the original computer solution should be saved in tabular form in the print-out listing. This table can then be used later with any propellant (and its design P_c) to establish corre-

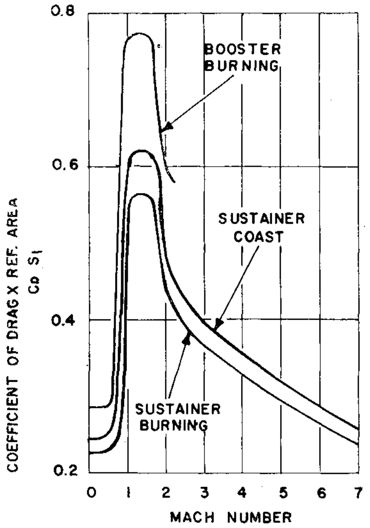

Fig. 1 C_DS_1 vs Mach number for vehicle-design 1.

sponding values for E and Q. If I_{s1}/I_s can be assumed to be a constant independent of time, then

$$\delta = (I_{s1} - I_s)/I_s \tag{31}$$

is also independent of time, and E can be determined exactly as

$$E = E_1E_1^{\delta} \tag{32}$$

If δ is also small, then

$$Q = Q_1(1 + \delta)(1 + E_1^{\delta})/2E_1^{\delta} \tag{33}$$

or

$$Q = Q_1(1 + \delta)(E_1^{\delta} - 1)/(\delta E_1^{\delta} \log_e E_1) \tag{34}$$

and a tabular listing of G_1I_{s1} and J_1I_{s1} vs time is not necessary to determine E and Q for new propellants or chamber pressures.

The variation in λ with P_c was not included explicitly in the preceding design equations; this correction can be made together with those in E and Q due to changes in $I_s = I_s(P_c)$. However, λ is also dependent on the thickness ratio of the combustion chamber t_c. This variation is not generally included, since designs 1 and 2 are to be of nearly the same shape. The following discussion is presented for those cases in solid propellant design where the variation in λ due to t_c is needed. This method can be replaced with empirical data of known designs or by plotting the results of several theoretical designs.

The inert motor weight W_i minus the nozzle weight is assumed proportional to that of the end closures and the outer cylindrical wall. For a constant strength of material, the wall and end closure thickness is assumed proportional to the product of P_c and D. Therefore, if D and L_c are the diameter and length of the combustion chamber, then

$$W_i = (C_1'D^2L_c + C_2'D^3)P_c/(1 - n)$$

where C_1' and C_2' are motor design constants, and n is the ratio of nozzle weight to W_i. Since the nozzle expansion ratio increases with P_c for a constant exhaust gas pressure, and W_i also increases with P_c, the fraction n is assumed to be insensitive to P_c for a given trajectory. The propellant weight W_p is proportional to the propellant volume (variations in density between propellants is neglected). The propellant volume is related to the chamber volume by the volume packing factor, and is therefore proportional to D^2L_c. Dividing the preceding equation by the propellant weight, and defining $t_c \equiv D/L_c$,

$$W_i/W_p = (C_1 + C_2t_c)P_c/(1 - n) \tag{35}$$

where C_1 and C_2 are new motor design constants. The ratio W_i/W_p is considered a function only of t_c and P_c. The propellant mass fraction is expressed by

$$1/\lambda = (1 + W_i/W_p) = 1 + P_c(C_1 + C_2t_c)/(1 - n) \tag{36}$$

For the case where empirical data are available, n need not be considered constant, and this equation shows that a plot of $1/\lambda$ vs t_c for various values of $P_c/(1 - n)$ would be useful. Assuming n and t_c are constant, changes in $1/\lambda$ with respect to P_c can be expressed in a form independent of the motor design constants C_1 and C_2:

$$\Delta(1/\lambda) = (1 - \lambda)(\Delta P_c)/P_c\lambda \tag{37}$$

Selection of Propellant and Chamber Pressure

The vehicle shape is not appreciably affected by the characteristics of a *liquid propellant*, so that the best propellant is simply that which has the largest I_s for a given P_c; on the other hand, the best P_c for this propellant is not so easily determined. Once the liquid propellant and exhaust gas pressure are chosen, $I_s(t)$ over the trajectory is known for each P_c, and the trajectory constants E and Q can be plotted vs P_c. If $\lambda(P_c)$ is known for the class of designs in question, then $K(P_c)$ and $\bar{K}(P_c)$ are also known functions. These can be used with the configuration constants f, B, or D to plot W_0 vs P_c for various P's. The optimum P_c is that which minimizes W_0 for a given P and configuration constraint.

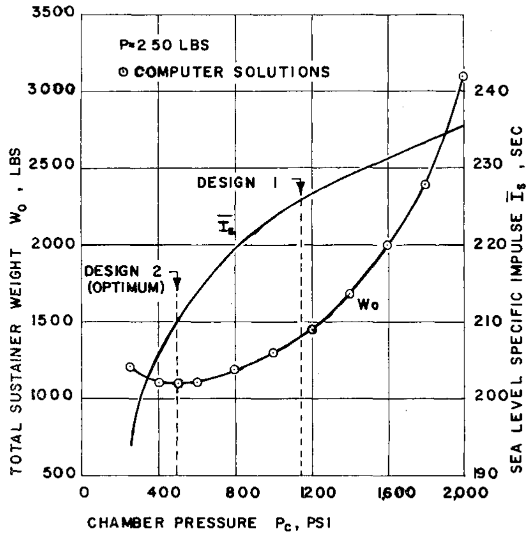

Fig. 2 Solution for optimum sustainer P

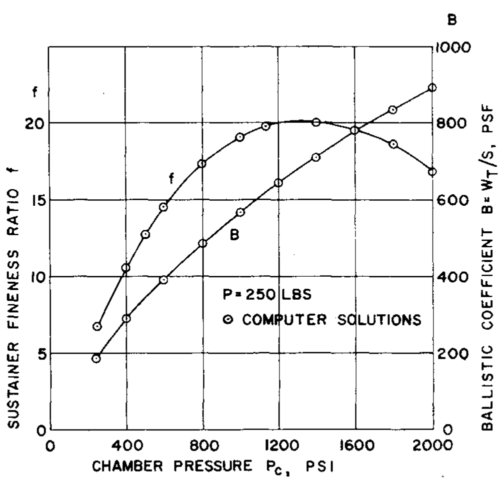

Fig. 3 f and B vs P_c for sustainer.

An approximate solution can be obtained by considering the specific impulse $\bar{I}_s$ at some selected altitude (or ambient pressure). Equations (31–33) and the value for $\bar{I}_{s1}$ are used to plot E and Q vs $\bar{I}_s$. The trajectory constants $K(\lambda, \bar{I}_s)$ and $\bar{K}(\lambda, \bar{I}_s)$ are now known functions. Therefore, for a given P, the configuration constraint f, B, or D can be used to plot W_0 vs $\bar{I}_s$ for various λ's. This graph relates the trajectory requirements prior to the consideration of a given propellant. For each propellant to be considered, λ vs $\bar{I}_s$ can be plotted from $\lambda(P_c)$ and $\bar{I}_s(P_c)$, so that a comparison of these graphs with that of the trajectory requirements can establish the optimum $\bar{I}_s$, and corresponding P_c, for a given payload and configuration constraint.

Among a given group of *solid propellants*, the best choice is not always obvious from a simple comparison of I_s. In the case of end burning, the propellant with the largest I_s may have a burning rate so small that a large body diameter is required in comparison to that for another propellant. Indeed, this large D may incur severe drag penalties that completely offset the advantages of high I_s. For each propellant, several chamber pressures should be selected, and E and Q should be plotted vs P_c. Since $\dot{r}(P_c)$ and $\lambda(P_c)$ are known, then b, K, and $\bar{K}$ are also known functions of P_c. By constraining b to the known value at each P_c, W_0 vs P_c can be plotted for various P's. Such graphs for each propellant will determine the proper choice of propellant and chamber pressure.

It is also possible to formulate approximate relations for solid propellants by using $\bar{I}_s$. If $\lambda(P_c)$ is known, then the trajectory requirements could be represented by plotting W_0 vs $\bar{I}_s$ for various $\dot{r}$'s, P's, and λ's. This graph can be compared with that of any propellant that shows $\dot{r}$ and $\bar{I}_s$ vs P_c.

It should be pointed out that there is often a minimum value for S_t expressed in Eq. (20). This is due to problems of nozzle erosion and nozzle plugging by pieces of the solid propellant. The so-called optimum P_c determined previously may be so large that S_t is prohibitively small; in this case, P_c should be decreased until an acceptable S_t is obtained in Eq. (20).

Example

A two-stage vehicle is considered for design 1. The second stage is the *Iris Sustainer Rocket,* which uses an end-burning solid propellant. Data for this stage were obtained from Refs. 4 and 5, and the drag data for both stages are shown in Fig. 1. The sustainer payload is 250 lb. (including 52.5 lb of structure and fins), and W_0 for the sustainer is 1411 lb. The booster payload is therefore 1411 lb, and other data for the booster are: $W_{R1} = 500$ lb; $W_{p1} = 330$ lb; $T = 2$ sec; $S_1 = 1.22$ ft²; $\dot{W}_{p1} =$ const; $I_{s1} = 225.8$ sec at $t = 0$ and 226.1 sec at $t = T$; and $\phi(t = 0) = 5°$. The propellant flow rates for both stages are assumed to be constant, and the trajectory is determined by a 7090 computer program that applies a two-stage two-dimensional point mass method with zero-lift, no-wind conditions for a spherical, nonrotating earth. Values for E_1 and Q_1 are determined for each stage by integrations over the trajectory. Various P_c's are chosen for the sustainer, and corresponding values of E and Q are determined by new integrations in a second computer program, which applies the design techniques of this paper.

Table 1 Comparison of sustainer designs 1 and 2[a]

Sustainer data	Design 1	Design 2
W_0, lb	1411	1112.2
P, lb	250.0	250.0
L_P, in.	80.0	63.75
L_R, in.	159.12	108.13
L_C, in.	152.35	102.36
S, ft²	0.797	1.000
T, sec	51.8	51.8
b, in.	148.0	98.0
λ	0.7898	0.8955
k	0.0296	0.0267
K,	4.383	3.357
$\bar{K}$, psf	395.87	372.71
P_c, psia	1140	500
Nozzle area ratio	10.07	5.45
Trajectory differences		
$\phi(t = 0)$, deg	5.85	5.85
Altitude ($t = 0$), ft	1259	1261
Velocity ($t = 0$), fps	1301	1301
Altitude ($t = T$), ft	166,869	157,234
Velocity ($t = T$), fps	6973	7035
Apogee, ft	934,623	934,639
Impact range, ft	633,320	670,253

[a] The payload for design 1 includes 52.5 lb of structure and fins.

The solution by this second program for W_0 and corresponding values for sea level $\bar{I}_{s1}$, f, and B are shown vs P_c in Figs. 2 and 3. Sustainer-design 2 corresponds to the P_c in Fig. 2, which minimizes the sustainer W_0. Variations in λ with P_c are estimated by using Eq. (37) with no corrections for t_c, and $k(P_c)$ is assumed to be a linear function. The Booster-design 2 is determined by Eq. (12), where W_0 is the two-stage weight, P is the Sustainer-design 2 weight (1112.2 lb), $S = 1.22$ ft², and K and $\bar{K}$ are 1.35 (no units) and 4.74 lb/ft², respectively. The design 2 vehicle is flown in the first computer program with constant $\dot{W}_p$ rather than the time function of Eq. (9). A comparison between Sustainer-designs 1 and 2 and differences between their trajectories are shown in Table 1. With the selection of a Sustainer-design 2, the Booster-design 2 can now be determined for arbitrary propellants, P_c's, and design constraints.

References

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